Slogan: equivariant version of Postnikov tower.

- Classical Postnikov tower

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P^n X \arrow[d] & \Sigma^n H(\pi_n X) \arrow[l] \\
X \arrow[ur] \arrow[r] & P^{n-1} X \arrow[d] & \Sigma^{n-1} H(\pi_{n-1} X) \arrow[l] \\
 & \vdots &
\end{tikzcd}
$$

$$\varprojlim P^n X \simeq X, \qquad \varinjlim P^n X \simeq *$$

satisfies $\pi_k P^n X = 0, \quad k > n.$

$\pi_k P^n X \cong \pi_k X, \quad k \leq n.$

Language $\tau_{\geq n+1}$ = full subcat obtained from $\{S^k : k > n\}$ by closing up under extension, cofiber, colimits

(not closed under limits & fibers !)

= full subcat of n-connective spectra.

$P^n X$ = Dror nullification w.r.t. $\tau_{\geq n+1}$.

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- Slice tower

1) $\mathrm{Map}^H(S^k, P^n X) = 0, \quad k > n$

$\mathrm{Map}^H(S^k, P^n X) = \mathrm{Map}^H(S^k, X), \quad k \leq n$

holds for all $H \leq G$. $X \in G\mathrm{Top}$ or $G\mathcal{SU}$.

2) Replace elets in $\tau_{\geq n+1}$ by $\{S^V\}$. — can choose what you want, but subject to defn.

3) Apply Dror nullification $P^n(-)$ to get a tower.

The desired Dror nullification functor should satisfy: $\forall X \in \mathrm{Top}$. X CW cpx, preferable cpt Hausdorff, then

- $X \longrightarrow P_{\mathcal{A}} X$ natural map.
- $P_{\mathcal{A}} X$ $\mathcal{A}$-null: $\forall A \in \mathcal{A}$ (all elets are cpt Hausdorff), $n \geq 0$,

$$[*, P_{\mathcal{A}} X] \xrightarrow{\cong} [\Sigma^n A, P_{\mathcal{A}} X].$$

- Z $\mathcal{A}$-null, $X \to Z$ map, $\exists$ lifting unique up to htpy

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \\ P_{\mathcal{A}} X \arrow[ur, dotted] & \end{tikzcd}$$

- $X \to Y \to Z$ htpy cofiber. $P_{\mathcal{A}} X$ contractible, then $P_{\mathcal{A}} Y \simeq P_{\mathcal{A}} Z$.
- Any X diagram. Then $P_{\mathcal{A}}(\mathrm{hocolim}_\alpha X_\alpha) \xrightarrow{\simeq} P_{\mathcal{A}}(\mathrm{hocolim}\, P_{\mathcal{A}} X_\alpha)$

• Candidate:

$P_{\mathcal{A}}$, $\mathcal{A} = \{S^W \wedge G/H_+ : W \geq V+1, H \leq G\}$.

$\Rightarrow$ all equivariant map $S^V \longrightarrow S^W$ are null.

won't change maps from S^V and smaller spheres

Prp 1) $X \to P_V X$ induces

$[S^{k,0} \wedge G/H_+ , -]_*$ $\begin{cases} \text{iso}, & 0 \le k \le \dim V^H \\ \text{epi}, & k = \dim V^H + 1. \end{cases}$

2) If $W \in R(G)$, $\dim W^H \le \dim V^H$ for all subgps $H \subseteq G$,

then $[S^W, X]_* \xrightarrow{\cong} [S^W, P_V X]$.

3) $K(\underline{\pi}_{V+1} X, V) \longrightarrow P_{V+1} X \longrightarrow P_V X$ htpy fib. seq.

4) $\operatorname{holim}(\cdots \longrightarrow P_{V+2} X \to P_{V+1} X \to P_V X) \simeq X$

5) If ρ regular rep of G, $\rho \subseteq V$, then

$P_V(S^V) \simeq K(\underline{A}, V)$. $\underline{A}$ Burnside-Mackey functor.

- Postnikov tower for $BU \times \mathbb{Z}$.

Consider $P_{n\mathbb{C}}(\mathbb{Z} \times BU) =: P_{(2n,n)}(\mathbb{Z} \times BU) =: P_{2n}(\mathbb{Z} \times BU)$

Thm $\beta : S^{2,1} \longrightarrow BU \times \mathbb{Z}$ Bott elet. i.e. $\beta \in \widetilde{KR}^{0,0}(S^{2,1})$.

$\beta^n : S^{2n,n} \longrightarrow BU \times \mathbb{Z}$.

Then $\exists$ htpy fib. seq.

$P_{2n}(S^{2n,n}) \xrightarrow{\beta^n} P_{2n}(\mathbb{Z} \times BU) \longrightarrow P_{2n-2}(\mathbb{Z} \times BU)$.

Thm $P_{2n}(S^{2n,n}) \simeq K(\mathbb{Z}(n), 2n)$

Cor Postnikov tower

$$\begin{tikzcd} & \vdots \arrow[d] \\ K(\mathbb{Z}(2),4) \arrow[r] & P_4(\mathbb{Z} \times BU) \arrow[d] \\ K(\mathbb{Z}(1),2) \arrow[r] & P_2(\mathbb{Z} \times BU) \end{tikzcd}$$

$$\begin{array}{ccc} & & \downarrow \\ \mathbb{Z} & \longrightarrow & P_0(\mathbb{Z}\times BU) \\ & & \downarrow \\ & & * \end{array}$$

holim (tower) $\simeq \mathbb{Z}\times BU$.

- $\pi_{2n,n}(\mathbb{Z}\times BU) = \underline{KR}^{2n,n}(pt) \cong \underline{\mathbb{Z}}$.

- Main Goal : htpy spectral sequence for X $\mathbb{Z}/2$ - space.

$$H^{p,-q/2}(X;\underline{\mathbb{Z}}) \Rightarrow [S^{-p-q,0}\wedge X_+,\ \mathbb{Z}\times BU]_*$$
$$= KR^{p+q,0}(X).$$

according to the previous corollary.

– Adams operations

Consider the diagram

$$\begin{tikzcd} S^{2n,n} \arrow[r, "\beta^n"] \arrow[d, "\cdot k^n"'] & \mathbb{Z}\times BU \arrow[d, "\psi^k"] \\ S^{2n,n} \arrow[r, "\beta^n"] & \mathbb{Z}\times BU \end{tikzcd}$$

— Adams operations

$\rightsquigarrow$

$$\begin{tikzcd} S^{2n,n} \arrow[r] \arrow[d, "\cdot k^n"'] & P_{2n}(S^{2n,n}) \arrow[r] \arrow[d, "P_{2n}(\cdot k^n)"] & P_{2n}(\mathbb{Z}\times BU) \arrow[d, "P_{2n}(\psi^k)"] \\ S^{2n,n} \arrow[r] & P_{2n}(S^{2n,n}) \arrow[r] & P_{2n}(\mathbb{Z}\times BU) \end{tikzcd}$$

and by

<u>FACT</u> $\quad P_{2n}(k) : P_{2n}(S^{2n,n}) \longrightarrow P_{2n}(S^{2n,n})$

$$\|$$

$\cdot k : K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(n), 2n)$

where k originally is the map $k : S^{2n,n} \to S^{2n,n}$

rep adding id to itself k times in $[S^{2n,n}, S^{2n,n}]_*$.

Then Adams operation on KR, $\psi^k : \mathbb{Z} \times BU \to \mathbb{Z} \times BU$.

induces Adams operation on F_n by $(\cdot k^n) : F_n \to F_n$, where

$K(\mathbb{Z}(n), 2n) \simeq F_n = \mathrm{hofib}(P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU))$.

$= P_{2n}(S^{2n,n})$.

- Rational tower

Note $H^{*,*}(BU) = H^{*,*}(pt)[c_1, c_2, \dots]$, $|c_i| = (2i, i)$.

$c_i : BU \to K(\mathbb{Z}(i), 2i)$

$\overset{P_n}{\rightsquigarrow}$ $P_n(BU) \longrightarrow P_{2n}(K(\mathbb{Z}(n), 2n)) = K(\mathbb{Z}(n), 2n)$

$\Rightarrow$ the composite of

$K(\mathbb{Z}(n), 2n) \simeq P_{2n}(S^{2n,n}) \xrightarrow{P_{2n}(\beta^n)} P_{2n}(\mathbb{Z} \times BU) \xrightarrow{c_n} K(\mathbb{Z}(n), 2n)$

is $(n-1)!$, since $c_n(\beta^n) = (n-1)! \cdot$ gen of $\tilde{H}^{2n,n}(S^{2n,n})$.

- Convergence

Conditionally converge. Converge for $p+q < 0$

- Stable version of previous s.s. : use connective cover of KR.

Want : cofib seq $\Sigma^{2,1} kr \xrightarrow{\beta} kr \to H\mathbb{Z}$

then Bockstein SS gives the desired SS.

First to understand kr. Let $W_n = \mathrm{hofib}(\mathbb{Z}\times BU \xrightarrow{\alpha} P_{2n-2}(\mathbb{Z}\times BU))$

Then apply $\Omega^{2,1}(-)$ to α. Note $X \in \mathcal{A}_{(2n-2,n-1)} \Rightarrow S^{2,1}\wedge X$ in $\mathcal{A}_{(2n,n)}$ $\Rightarrow \Omega^{2,1}P_{2n}(\mathbb{Z}\times BU) \in \mathcal{A}_{(2n-2,n-1)}$ since $P_{2n}(\mathbb{Z}\times BU)$ is $\mathcal{A}_{(2n,n)}$-null. So $\exists$ lift unique up to htpy

$$\begin{tikzcd}
\Omega^{2,1}(\mathbb{Z}\times BU) \arrow[r, "\Omega^{2,1}\alpha"] \arrow[d] & \Omega^{2,1}P_{2n}(\mathbb{Z}\times BU) \\
P_{2n-2}\Omega^{2,1}(\mathbb{Z}\times BU) \arrow[ur, dashed, "\ell"'] &
\end{tikzcd}$$

Let $\beta : \mathbb{Z}\times BU \longrightarrow \Omega^{2,1}(\mathbb{Z}\times BU)$ Bott map, get

$$\begin{tikzcd}
W_n \arrow[r] \arrow[dd, dashed, "\exists\sigma"'] & \mathbb{Z}\times BU \arrow[r] \arrow[d, "\beta"] & P_{2n-2}(\mathbb{Z}\times BU) \arrow[d, "P_{2n-2}\beta"] \\
& \Omega^{2,1}(\mathbb{Z}\times BU) \arrow[r] \arrow[d, "id"] & P_{2n-2}(\Omega^{2,1}(\mathbb{Z}\times BU)) \arrow[d, "\ell"] \\
\Omega^{2,1}W_{n+1} \arrow[r] & \Omega^{2,1}(\mathbb{Z}\times BU) \arrow[r] & \Omega^{2,1}P_{2n-2}(\mathbb{Z}\times BU)
\end{tikzcd}$$

then $\exists\sigma : W_n \to \Omega^{2,1}W_{n+1}$ makes diagram commute. Use the following lemma to conclude σ is w.e.:

<u>Lem</u> $X, Y \in C_2\text{-space}_*$ and

$$[S^{k,0}, X]_* = [S^{k,0}, Y]_* = 0, \quad 0 \le k < n$$

$$[\mathbb{Z}/2_+ \wedge S^{k,0}, X]_* = [\mathbb{Z}/2_+ \wedge S^{k,0}, Y]_* = 0, \quad 0 \le k < 2n.$$

Then $X \to Y$ w.e. $\Leftrightarrow$ it induces iso for $k \ge 0$

$$[S^{2n+k,n}, X]_* \cong [S^{2n+k,n}, Y]_*$$

$$[\mathbb{Z}/2_+ \wedge S^{2n+k,n}, X] \cong [\mathbb{Z}/2_+ \wedge S^{2n+k,n}, Y]_*.$$

This is also used to show $F_n \simeq K(\mathbb{Z}(n), 2n)$.

Def $kr = \{W_n,\ W_n \to \Omega^{2,1} W_{n+1}\}$

connective KR spectrum.

$\Rightarrow$ tower

$$\begin{tikzcd}
\vdots \arrow[d] & \\
\Sigma^{2,1} kr \arrow[r] \arrow[d, "\beta"] & \Sigma^{2,1} H\mathbb{Z} \\
kr \arrow[r] \arrow[d, "\Sigma^{-2,-1}\beta"] & H\mathbb{Z} \\
\Sigma^{-2,-1} kr \arrow[r] \arrow[d] & \Sigma^{-2,-1} H\mathbb{Z} \\
\vdots &
\end{tikzcd}$$

colim $=$ KR

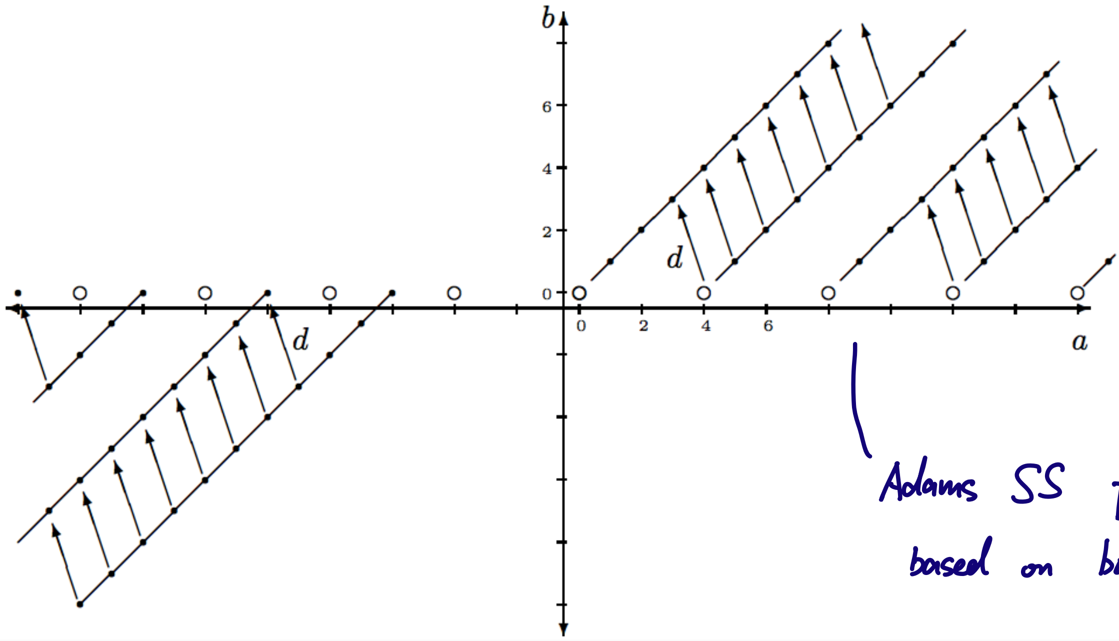
lim $= *$

$\Rightarrow$ $H^{p,-\frac{q}{2}}(X;\mathbb{Z}) \Rightarrow KR^{p+q,0}(X)$. converges conditionally

It can be used to compute htpy of $P_n(\mathbb{Z}\times BU)$, and so W_n.

So it can be used to determine $kr^{*,*}(pt)$. This following is the S.S. for $X = pt$:

$$H^{p,-q/2}(pt;\mathbb{Z}) \Rightarrow KR^{p+q,0}(pt) = KO^{p+q}(pt)$$



Adams SS for bo based on bu.

(a.b) : $H^{b,\frac{a+b}{2}}(pt)$

line a = N : associated graded of KO^{-N}

This SS collapse at next page